Generalized Dirichlet to Neumann operator on invariant differential forms and equivariant cohomology

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Abstract

In [6], Belishev and Sharafutdinov consider a compact Riemannian manifold M with boundary ∂M. They define a generalized Dirichlet to Neumann (DN) operator Λ on all forms on the boundary and they prove that the real additive de Rham cohomology structure of the manifold in question is completely determined by Λ. This shows that the DN map Λ inscribes into the list of objects of algebraic topology. In this paper, we suppose G is a torus acting by isometries on M. Given X in the Lie algebra of G and the corresponding vector field X_M on M, one defines Witten's inhomogeneous coboundary operator $\mathrm{d}_{X_M} = \mathrm{d} + \iota_{X_M}$ on invariant forms on M. The main purpose is to adapt Belishev and Sharafutdinov's boundary data to invariant forms in terms of the operator d_{X_M} and its adjoint δ_{X_M}. In other words, we define an operator Λ_{X_M} on invariant forms on the boundary which we call the X_M-DN map and using this we recover the long exact X_M-cohomology sequence of the topological pair $(M, \partial M)$ from an isomorphism with the long exact sequence formed from our boundary data. We then show that Λ_{X_M} completely determines the free part of the relative and absolute equivariant cohomology groups of M when the set of zeros of the corresponding vector field X_M is equal to the fixed point set F for the G-action. In addition, we partially determine the mixed cup product (the ring structure) of X_M-cohomology groups from Λ_{X_M}. These results explain to what extent the equivariant topology of the manifold in question is determined by the X_M-DN map Λ_{X_M}. Finally, we illustrate the connection between Belishev and Sharafutdinov's boundary data on ∂F and ours on ∂M.

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1 Introduction

The classical Dirichlet-to-Neumann (DN) operator $\Lambda_{cl} : C^\infty(\partial M) \longrightarrow C^\infty(\partial M)$ is defined by $\Lambda_{cl}\theta = \partial\omega/\partial\nu$, where ω is the solution to the Dirichlet problem

$$\Delta\omega = 0, \quad \omega\,|_{\partial M} = \theta$$

and ν is the unit outer normal to the boundary. In the scope of inverse problems of reconstructing a manifold from the boundary measurements, the following question is of great theoretical and applied interest [6]:

"To what extent are the topology and geometry of M determined by the Dirichlet-to-Neumann map"?

In this paper we are interested in the topology aspect while the geometry aspect of the above question has been studied in [12] and [10].

Much effort has been made to answer the topology aspect of this question. For instance, in the case of a two-dimensional manifold M with a connected boundary, an explicit formula is obtained which expresses the Euler characteristic of M in terms of Λ_{cl} where the Euler characteristic completely determines the topology of M in this case [5]. In the three-dimensional case [4], some formulas are obtained which express the Betti numbers $\beta_1(M)$ and $\beta_2(M)$ in terms of Λ_{cl} and $\overrightarrow{\Lambda} : C^\infty(T(\partial M)) \longrightarrow C^\infty(T(\partial M))$.

For more topological aspects, Belishev and Sharafutdinov [6] prove that the real additive de Rham cohomology of a compact, connected, oriented smooth Riemannian manifold M of dimension n with boundary is completely determined by its boundary data $(\partial M, \Lambda)$ where $\Lambda : \Omega^k(\partial M) \longrightarrow \Omega^{n-k-1}(\partial M)$ is a generalization of the classical Dirichlet-to-Neumann operator Λ_{cl} to the space of differential forms. More precisely, they define the DN operator Λ as follows [6]: given $\theta \in \Omega^k(\partial M)$, the boundary value problem

$$\Delta\omega = 0, \quad i^*\omega = \theta, \quad i^*(\delta\omega) = 0$$

is solvable and the operator Λ is given by the formula

$$\Lambda\theta = i^*(\star \mathrm{d}\omega)$$

where i^* is the pullback by the inclusion map $i : \partial M \hookrightarrow M$. Here δ is the formal adjoint of d relative to the L^2-inner product

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge (\star\beta)$$

which is defined on $\Omega^k(M)$, and $\star : \Omega^k \to \Omega^{n-k}$ is the Hodge star operator.

More concretely, there are two distinguished finite dimensional subspaces of $\mathcal{H}^k(M) = \ker \mathrm{d} \cap \ker \delta$, whose elements are called Dirichlet and Neumann harmonic fields respectively, namely

$$\mathcal{H}^k_D(M) = \{\lambda \in \mathcal{H}^k(M) \mid i^*\lambda = 0\}, \quad \mathcal{H}^k_N(M) = \{\lambda \in \mathcal{H}^k(M) \mid i^* \star \lambda = 0\}.$$

The dimensions of these spaces are given by

$$\dim \mathcal{H}^k_D(M) = \dim \mathcal{H}^{n-k}_N(M) = \beta_k(M)$$

where $\beta_k(M)$ is the kth Betti number [14] . They prove the following theorem

Theorem 1.1 (Belishev-Sharafutdinov [6]) *For any $0 \le k \le n-1$, the range of the operator*

$$\Lambda + (-1)^{nk+k+n}\mathrm{d}\Lambda^{-1}\mathrm{d} : \Omega^k(\partial M) \longrightarrow \Omega^{n-k-1}(\partial M)$$

is $i^*\mathcal{H}^{n-k-1}_N(M)$.

But $i^*\mathcal{H}^k_N(M) \cong \mathcal{H}^k_N(M) \cong H^k(M)$. Hence, $(\Lambda + (-1)^{nk+k+1}\mathrm{d}\Lambda^{-1}\mathrm{d})\Omega^{n-k-1}(\partial M) \cong H^k(M) \cong \mathcal{H}^k_N(M)$. Using, Poincaré-Lefscetz duality, $H^k(M) \cong H^{n-k}(M, \partial M)$. So the above theorem immediately implies that the data $(\partial M, \Lambda)$ determines the absolute and relative de Rham cohomology groups.

In addition, they present the following theorem which gives the lower bound for the Betti numbers of the manifold M through the DN operator Λ.

Theorem 1.2 (Belishev-Sharafutdinov [6]) *The kernel* $\ker\Lambda$ *contains the space $\mathcal{E}^k(\partial M)$ of exact forms and*

$$\dim[\ker\Lambda^k/\mathcal{E}^k(\partial M)] \le \min\{\beta_k(\partial M), \beta_k(M)\}$$

where $\beta_k(\partial M)$ and $\beta_k(M)$ are the Betti numbers, and Λ^k is the restriction of Λ to $\Omega^k(\partial M)$.

But at the end of their paper, they posed the following topological open problem:

"Can the multiplicative structure of cohomologies be recovered from our data $(\partial M,\Lambda)$*?"*.

In 2009, Shonkwiler in [16] gave a partial answer to the above question. He presents a well-defined map which is

$$(\phi,\psi)\longmapsto \Lambda((-1)^k\phi\wedge\Lambda^{-1}\psi),\quad \forall(\phi,\psi)\in i^*\mathcal{H}^k_N(M)\times i^*\star\mathcal{H}^l_D(M) \tag{1.1}$$

and then uses it to give a partial answer to that question. More precisely, by using the classical wedge product between the differential forms, he considers the mixed cup product between the absolute cohomology $H^k(M,\mathbb{R})$ and the relative cohomology $H^l(M,\partial M,\mathbb{R})$, i.e.

$$\cup: H^k(M,\mathbb{R})\times H^l(M,\partial M,\mathbb{R})\longrightarrow H^{k+l}(M,\partial M,\mathbb{R})$$

and then he restricts $H^l(M,\partial M,\mathbb{R})$ to come from the boundary subspace which is defined by DeTurck and Gluck [8] as the subspace of exact forms which satisfy the Dirichlet boundary condition (i.e. i^* of these exact forms are zero) and then he presents the following theorem as a partial answer to Belishev and Sharafutdinov's question:

Theorem 1.3 (Shonkwiler [16]) *The boundary data* $(\partial M,\Lambda)$ *completely determines the mixed cup product in terms of the map (1.1) when the relative cohomology class is restricted to come from the boundary subspace.*

From another hand, in [1], we consider a compact, oriented, smooth Riemannian manifold M with boundary and we suppose G is a torus acting by isometries on M and denote by Ω^k_G the k-forms invarient under action of G. Given X in the Lie algebra of G and corresponding vector field X_M on M, we consider Witten's coboundary operator $\mathsf{d}_{X_M}=\mathsf{d}+\iota_{X_M}$. This operator is no longer homogeneous in the degree of the smooth invariant form on M: if $\omega\in\Omega^k_G$ then $\mathsf{d}_{X_M}\omega\in\Omega^{k+1}_G\oplus\Omega^{k-1}_G$. Note then that $\mathsf{d}_{X_M}:\Omega^{\pm}_G\to\Omega^{\mp}_G$, where $\Omega^{\pm}_G$ is the space of invariant forms of even $(+)$ or odd $(-)$ degree. Let δ_{X_M} be the adjoint of d_{X_M} and the resulting *Witten-Hodge-Laplacian* is $\Delta_{X_M}=(\mathsf{d}_{X_M}+\delta_{X_M})^2=\mathsf{d}_{X_M}\delta_{X_M}+\delta_{X_M}\mathsf{d}_{X_M}$.

Because the forms are invariant, it is easy to see that $\mathsf{d}^2_{X_M}=0$ (see [1] for details). In this setting, we define two types of X_M-cohomology, the absolute X_M-cohomology $H^{\pm}_{X_M}(M)$ and the relative X_M-cohomology $H^{\pm}_{X_M}(M,\partial M)$. The first is the cohomology of the complex $(\Omega_G,\,\mathsf{d}_{X_M})$, while the second is the cohomology of the subcomplex $(\Omega_{G,D},\,\mathsf{d}_{X_M})$, where $\omega\in\Omega^{\pm}_{G,D}$ if it satisfies $i^*\omega=0$ (the D is for Dirichlet boundary condition). One also defines $\Omega^{\pm}_{G,N}(M)=\left\{\alpha\in\Omega^{\pm}_G(M)\mid i^*(\star\alpha)=0\right\}$ (Neumann boundary condition). Clearly, the Hodge star provides an isomorphism

$$\star:\Omega^{\pm}_{G,D}\xrightarrow{\sim}\Omega^{n-\pm}_{G,N}$$

where we write $n-\pm$ for the parity (modulo 2) resulting from subtracting an even/odd number from n. Furthermore, because d_{X_M} and i^* commute, it follows that d_{X_M} preserves Dirichlet boundary conditions while δ_{X_M} preserves Neumann boundary conditions. Because of boundary terms, the null space of Δ_{X_M} no longer coincides with the closed and co-closed forms in Witten sense. Elements of $\ker\Delta_{X_M}$ are called X_M-*harmonic forms* while ω which satisfy $\mathsf{d}_{X_M}\omega=\delta_{X_M}\omega=0$ are X_M-*harmonic fields* (following [1]); it is clear that every X_M-harmonic field is an X_M-harmonic form, but the converse is false. The space of X_M-harmonic *fields* is denoted $\mathcal{H}^{\pm}_{X_M}(M)$ (so $\mathcal{H}^{*}_{X_M}(M)\subset\ker\Delta_{X_M}$). In fact, the space $\mathcal{H}^{\pm}_{X_M}(M)$ is infinite dimensional and so is much too big to represent the X_M-cohomology, hence, we restrict $\mathcal{H}^{\pm}_{X_M}(M)$ into each of two finite dimensional subspaces, namely $\mathcal{H}^{\pm}_{X_M,D}(M)$ and $\mathcal{H}^{\pm}_{X_M,N}(M)$ with the obvious meanings (Dirichlet and Neumann X_M-harmonic fields, respectively). There are therefore two

different candidates for X_M-harmonic representatives when the boundary is present. This construction firstly leads us to present the X_M-Hodge-Morrey decomposition theorem which states that

$$\Omega^{\pm}_G(M) = \mathcal{E}^{\pm}_{X_M}(M) \oplus \mathcal{C}^{\pm}_{X_M}(M) \oplus \mathcal{H}^{\pm}_{X_M}(M) \tag{1.2}$$

where $\mathcal{E}^{\pm}_{X_M}(M) = \{\mathrm{d}_{X_M}\alpha \mid \alpha \in \Omega^{\mp}_{G,D}\}$ and $\mathcal{C}^{\pm}_{X_M}(M) = \{\delta_{X_M}\beta \mid \beta \in \Omega^{\mp}_{G,N}\}$. This decomposition is orthogonal with respect to the L^2-inner product given above.

In addition, in [1] we present X_M-Friedrichs Decomposition Theorem which states that

$$\begin{aligned} \mathcal{H}^{\pm}_{X_M}(M) &= \mathcal{H}^{\pm}_{X_M,D}(M) \oplus \mathcal{H}^{\pm}_{X_M,\mathrm{co}}(M) && (1.3)\\ \mathcal{H}^{\pm}_{X_M}(M) &= \mathcal{H}^{\pm}_{X_M,N}(M) \oplus \mathcal{H}^{\pm}_{X_M,\mathrm{ex}}(M) && (1.4) \end{aligned}$$

where $\mathcal{H}^{\pm}_{X_M,\mathrm{ex}}(M) = \{\xi \in \mathcal{H}^{\pm}_{X_M}(M) \mid \xi = \mathrm{d}_{X_M}\sigma\}$ and $\mathcal{H}^{\pm}_{X_M,\mathrm{co}}(M) = \{\eta \in \mathcal{H}^{\pm}_{X_M}(M) \mid \eta = \delta_{X_M}\alpha\}$. These give the orthogonal *X_M-Hodge-Morrey-Friedrichs* [1] decompositions,

$$\begin{aligned} \Omega^{\pm}_G(M) &= \mathcal{E}^{\pm}_{X_M}(M) \oplus \mathcal{C}^{\pm}_{X_M}(M) \oplus \mathcal{H}^{\pm}_{X_M,D}(M) \oplus \mathcal{H}^{\pm}_{X_M,\mathrm{co}}(M) \\ &= \mathcal{E}^{\pm}_{X_M}(M) \oplus \mathcal{C}^{\pm}_{X_M}(M) \oplus \mathcal{H}^{\pm}_{X_M,N}(M) \oplus \mathcal{H}^{\pm}_{X_M,\mathrm{ex}}(M) && (1.5) \end{aligned}$$

The two decompositions are related by the Hodge star operator. The orthogonality of (1.2-1.5) follows from Green's formula for d_{X_M} and δ_{X_M} of [1] which states

$$\langle \mathrm{d}_{X_M}\alpha, \beta\rangle = \langle \alpha, \delta_{X_M}\beta\rangle + \int_{\partial M} i^*(\alpha \wedge \star\beta) \tag{1.6}$$

for all $\alpha, \beta \in \Omega_G$.

The consequence for X_M-cohomology is that each class in $H^{\pm}_{X_M}(M)$ is represented by a unique X_M-harmonic field in $\mathcal{H}^{\pm}_{X_M,N}(M)$, and each relative class in $H^{\pm}_{X_M}(M,\partial M)$ is represented by a unique X_M-harmonic field in $\mathcal{H}^{\pm}_{X_M,D}(M)$. We also elucidate the connection between the X_M-cohomology groups and the relative and absolute equivariant cohomology groups.

Our construction of the *X_M-Hodge-Morrey-Friedrichs* decompositions (1.5) of smooth invariant differential forms gives us insight to create boundary data which is a generalization of Belishev and Sharafutdinov's boundary data on $\Omega^{\pm}_G(\partial M)$.

In this paper, we take a more topological approach, looking to determine the X_M-cohomology groups and the free part of the equivariant cohomology groups from the generalized boundary data. To this end, we need first in section 2 to prove that our concrete realizations $\mathcal{H}^{\pm}_{X_M,N}(M)$ and $\mathcal{H}^{\pm}_{X_M,D}(M)$ of the absolute and relative X_M-cohomology groups respectively meet only at the origin while in section 3 we define the X_M-DN operator Λ_{X_M} on $\Omega^{\pm}_G(\partial M)$, the definition involves showing that certain boundary value problems are solvable. Our definition of Λ_{X_M} represents a generalization of Belishev and Sharafutdinov's DN-operator Λ on $\Omega^{\pm}_G(\partial M)$ in the sense that when $X_M = 0$, we would get $\Lambda_0 = \Lambda$. Finally, in the remaining sections, we explain to what extent the equivariant topology of the manifold in question is determined by the X_M-DN map Λ_{X_M}.

2 Main results

We consider a compact, connected, oriented, smooth Riemannian manifold M with boundary and we suppose G is a torus acting by isometries on M. Given X in the Lie algebra and corresponding vector

field X_M on M, one defines Witten's inhomogeneous coboundary operator $\mathsf{d}_{X_M} = \mathsf{d} + \iota_{X_M} : \Omega_G^{\pm} \to \Omega_G^{\mp}$ and the resulting X_M-harmonic fields and forms as described in the introduction.

We introduce the following definitions of the X_M-trace spaces

$$i^*\mathcal{H}^{\pm}_{X_M}(M) = \{i^*\lambda \mid \lambda \in \mathcal{H}^{\pm}_{X_M}(M)\}, \quad i^*\mathcal{H}^{\pm}_{X_M,N}(M) = \{i^*\lambda_N \mid \lambda_N \in \mathcal{H}^{\pm}_{X_M,N}(M)\}.$$

we call $i^*\mathcal{H}^{\pm}_{X_M,N}(M)$ the Neumann X_M-trace space.

Remark 2.1 Along the boundary of M, any smooth differential form ω has a natural decomposition into tangential ($t\omega$) and normal($n\omega$) components. i.e.

$$\omega\mid_{\partial M} = t\omega + n\omega$$

and the tangential component $t\omega$ is uniquely determined by the pull-back $i^*\omega$ and it has been denoted in a slight abuse of notation by $i^*\omega = i^*t\omega = t\omega$. The normal and tangential components of ω are Hodge adjoint to each other [14], i.e.

$$\star(n\omega) = t(\star\omega) = i^* \star \omega.$$

In order to prove Theorem 2.3, we will use the strong unique continuation theorem, due to Aronszajn [2], Aronszajn, Krzywicki and Szarski [3]. In [11], Kazdan writes this theorem in terms of Laplacian operator Δ but he mentions that it is still valid for any operator having the diagonal form $P = \Delta I +$ lower-order terms, where I is the identity matrix. Hence, one can state this theorem in terms of diagonal form operator by the following form:

Theorem 2.2 (Strong Unique Continuation Theorem [11]) *Let $\overline{M}$ be a Riemannian manifold with Lipschitz continuous metric, and let ω be a differential form having first derivatives in L^2 that satisfies $P(\omega) = 0$ where P is a diagonal form operator. If ω has a zero of infinite order at some point in $\overline{M}$, then ω is identically zero on $\overline{M}$.*

Now, we are ready to present our main results.

Theorem 2.3 *Let M be a compact, connected, oriented smooth Riemannian manifold of dimension n with boundary and with an action of a torus G which acts by isometries on M. If an X_M-harmonic field $\lambda \in \mathcal{H}^{\pm}_{X_M}(M)$ vanishes on the boundary ∂M, then $\lambda \equiv 0$, i.e.*

$$\mathcal{H}^{\pm}_{X_M,N}(M) \cap \mathcal{H}^{\pm}_{X_M,D}(M) = \{0\} \tag{2.1}$$

PROOF: Suppose $\lambda \in \mathcal{H}^{\pm}_{X_M,N}(M) \cap \mathcal{H}^{\pm}_{X_M,D}(M)$, then λ is smooth by theorem 3.4(c) of [1]. Since $i^*\lambda = i^*\star\lambda = 0$ then remark 2.1 asserts that $t\lambda = n\lambda = 0$. Hence $\lambda\mid_{\partial M} \equiv 0$ and we get that $(\iota_{X_M}\lambda)\mid_{\partial M} = 0$ as well.

The proof is local so we can consider M to be the upper half space in $\mathbb{R}^n$ with $\partial M = \mathbb{R}^{n-1}$. Since the metric, the differential form λ and the vector field X_M are given in the upper half space, we can extend them from there to all of $\mathbb{R}^n$ by reflection in $\partial M = \mathbb{R}^{n-1}$. The resulting objects are: the extended metric, which will be Lipschitz continuous [7]; we extend λ to all of $\mathbb{R}^n$ by making it odd with respect to reflection in $\mathbb{R}^{n-1}$ and extend X_M to all of $\mathbb{R}^n$ by making it even with respect to reflection in $\mathbb{R}^{n-1}$ and extended X_M will be a Lipschitz continuous vector field. But the original λ satisfies $\lambda\mid_{\partial M} \equiv 0$ and $\mathsf{d}_{X_M}\lambda = \delta_{X_M}\lambda = 0$ on $\mathbb{R}^{n-1}$, hence the extended one will be of class C^1 and satisfy $\mathsf{d}_{X_M}\lambda = \delta_{X_M}\lambda = 0$ on $\mathbb{R}^n$, i.e. the extended λ satisfies $P(\lambda) = \Delta_{X_M}\lambda = 0$ on all of $\mathbb{R}^n$ where the operator Δ_{X_M} has diagonal

form, i.e. $P = \Delta_{X_M} = \Delta I +$ lower-order terms, and I is the identity matrix. So far, we satisfy the first condition of theorem 2.2.

Now, we need to satisfy the remaining hypotheses of theorem 2.2. Let $x = (x', x_n) = (x_1, x_2, ..., x_{n-1}, x_n)$ be a coordinates chart where $x' = (x_1, x_2, ..., x_{n-1})$ is a chart on the boundary ∂M and x_n is the distance to the boundary. In these coordinates $x_n > 0$ in M and ∂M is locally characterized by $x_n = 0$. These coordinates are called boundary normal coordinates and the Riemannian metric in these coordinates has the form $\sum_{m,r=1}^{n-1} h_{m,r}(x) dx^m \otimes dx^r + dx^n \otimes dx^n$.

Now, we consider a neighborhood of $p \in \partial M$ where our boundary normal coordinates are well defined. We can write $\lambda = \alpha + \beta \wedge dx_n$ where $\alpha = \Sigma f_I(x) dx^I$, $\beta = \Sigma g_I(x) dx^I$ and $I \subset \{1, 2, ..., n-1\}$. Our goal is to prove that all the partial derivatives of the coefficients of λ (i.e. $f_I(x)$ and $g_I(x)$) vanish at $p \in \partial M$. Now, $\lambda \mid_{\partial M} \equiv 0$ which implies that $f_I(x', 0) = g_I(x', 0) = 0$. Hence, we can apply Hadamard's lemma to $f_I(x)$ and $g_I(x)$ and deduce that $f_I(x) = x_n \overline{f_I}(x)$ and $g_I(x) = x_n \overline{g_I}(x)$ for some smooth functions $\overline{f_I}(x)$ and $\overline{g_I}(x)$. Moreover, these representations for $f_I(x)$ and $g_I(x)$ help us to conclude that all the higher partial derivatives of $f_I(x)$ and $g_I(x)$ with respect to the coordinates of x' (i.e. except the normal direction coordinate x_n) at the point p are all zero. i.e.

$$\frac{\partial^{|s|} f_I(x', 0)}{\partial x_1^{s_1} ... \partial x_{n-1}^{s_{n-1}}} = \frac{\partial^{|s|} g_I(x', 0)}{\partial x_1^{s_1} ... \partial x_{n-1}^{s_{n-1}}} = 0, \quad \forall s_1, s_2, ..., s_{n-1} = 0, 1, 2, ...$$

Therefore, we only need to prove that all the higher partial derivatives of $f_I(x)$ and $g_I(x)$ in the normal direction are zero to deduce that the Taylor series of $f_I(x)$ and $g_I(x)$ around $x_n = 0$ are zero.

For contradiction, suppose the Taylor series of $f_I(x)$ and $g_I(x)$ around $x_n = 0$ are not zero at $p \in \partial M$ which means that there exist the largest positive integer numbers k and j such that $f_I(x) = x_n^k \widehat{f_I}(x)$ and $g_I(x) = x_n^j \widehat{g_J}(x)$ where $\widehat{f_I}(x', 0) \neq 0$ and $\widehat{g_J}(x', 0) \neq 0$ for some I, J. Thus, we can always write λ in the following form $\lambda = x_n^k \tau + x_n^j \rho \wedge dx_n$ where the differential forms τ and ρ do not contain dx_n. Applying $d_{X_M} \lambda = 0$, we get

$$0 = d_{X_M} \lambda = k x_n^{k-1} dx_n \wedge \tau + x_n^k d\tau + x_n^j d\rho \wedge dx_n + x_n^k \iota_{X_M} \tau + x_n^j \iota_{X_M} (\rho \wedge dx_n).$$

Now, reducing this equation modulo x_n^k we conclude that the term $x_n^j (d\rho \wedge dx_n + \iota_{X_M}(\rho \wedge dx_n)) \not\equiv 0$ modulo x_n^k because the term $k x_n^{k-1} dx_n \wedge \tau \not\equiv 0$ modulo x_n^k and as a consequence, we infer that $k > j$.

Similarly, we can calculate $\delta_{X_M} \lambda = -(\mp)^n (\star d \star \lambda + \star \iota_{X_M} \star \lambda) = 0$ (using the Riemannian metric above). For simplicity, it is enough to calculate $d \star \lambda + \iota_{X_M} \star \lambda = 0$ where $\star \lambda = x_n^k \xi \wedge dx_n + x_n^j \zeta$ such that the differential forms ξ and ζ do not contain dx_n and both of them should contain many of the coefficients $h_{m,r}(x)$. Hence, we get

$$0 = d \star \lambda + \iota_{X_M} \star \lambda = x_n^k d\xi \wedge dx_n + j x_n^{j-1} dx_n \wedge \zeta + x_n^j d\zeta + x_n^k \iota_{X_M} (\xi \wedge dx_n) + x_n^j \iota_{X_M} \zeta.$$

Reducing this equation modulo x_n^j and for the same reason above but replacing k by j, then we can infer that $k < j$, but this is a contradiction, then there are not such largest positive integer numbers k and j. Hence, the Taylor series for the coefficients $f_I(x)$ and $g_I(x)$ around $x_n = 0$ must be zero at $p \in \partial M$, i.e.

$$\frac{\partial^r f_I(x', 0)}{\partial x_n^r} = \frac{\partial^r g_I(x', 0)}{\partial x_n^r} = 0, \quad \forall r = 0, 1, 2, \ldots$$

It means that all the higher partial derivatives of $f_I(x)$ and $g_I(x)$ we have already considered vanish at all points of the boundary ∂M. Thus, this facts are enough to show the mixed partial derivatives including x_n also vanish at the boundary. Hence, λ has a zero of infinite order at $p \in \partial M$.

The remaining possibility of one of the the coefficients f_I and g_I having finite order and the other infinite order in x_n follows from the same argument as above.

Thus, λ satisfies all the hypotheses of the strong Unique Continuation Theorem 2.2 then λ must be zero on all of $\mathbb{R}^n$. Since M is assumed to be connected, λ must be identically zero on all of M, i.e. $\lambda \equiv 0$. ❒

As a consequence of Theorem 2.3, we obtain the following results.

Corollary 2.4

$$\mathcal{H}^{\pm}_{X_M}(M) = \mathcal{H}^{\pm}_{X_M,\mathrm{ex}}(M) + \mathcal{H}^{\pm}_{X_M,\mathrm{co}}(M) \tag{2.2}$$

where " $+$ " is not direct sum.

PROOF: The X_M-Friedrichs Decomposition Theorem (1.3 and 1.4) shows that $(\mathcal{H}^{\pm}_{X_M,D}(M))^{\perp} \cap \mathcal{H}^{\pm}_{X_M}(M) = \mathcal{H}^{\pm}_{X_M,\mathrm{co}}(M)$ and $(\mathcal{H}^{\pm}_{X_M,N}(M))^{\perp} \cap \mathcal{H}^{\pm}_{X_M}(M) = \mathcal{H}^{\pm}_{X_M,\mathrm{ex}}(M)$. Hence, using these facts together with Theorem 2.3, we conclude eq.(2.2.) ❒

Corollary 2.5 *The trace map $i^* : \mathcal{H}^{\pm}_{X_M,N}(M) \longrightarrow i^*\mathcal{H}^{\pm}_{X_M,N}(M)$ defines an isomorphism.*

PROOF: It is clear that i^* is surjective and we can use theorem 2.3 to prove the kernel of the linear map i^* is zero (i.e. $\ker i^* = \{0\}$) which implies that i^* is injective. Thus, i^* is bijection. ❒

Corollary 2.6 *1- The map $f : i^*\mathcal{H}^{\pm}_{X_M,N}(M) \longrightarrow H^{\pm}_{X_M}(M)$ defined by $f(i^*\lambda_N) = [\lambda_N]$ for $\lambda_N \in \mathcal{H}^{\pm}_{X_M,N}(M)$ is an isomorphism.*

2- The map $h : i^*\mathcal{H}^{n-\pm}_{X_M,N}(M) \longrightarrow H^{\pm}_{X_M}(M,\partial M)$ defined by $h(i^*\lambda_N) = [\star\lambda_N]$ for $\lambda_N \in \mathcal{H}^{n-\pm}_{X_M,N}(M)$ is an isomorphism.

PROOF:

1- f is a well-defined map because $\ker i^* = \{0\}$ (corollary 2.5). Furthermore, f is a bijection because there exists a unique Neumann X_M-harmonic field in any absolute X_M- cohomology class (Theorem 3.16(a) of [1]) hence part (1) holds.

2- It follows from part (1) by using X_M-Poincaré-Lefschetz duality (Theorem 3.16(c) of [1]).

❒

Corollary 2.7 $\dim(\mathcal{H}^{\pm}_{X_M,N}(M)) = \dim(i^*\mathcal{H}^{\pm}_{X_M,N}(M)) = \dim(H^{\pm}_{X_M}(M)) = \dim(H^{n-\pm}_{X_M}(M,\partial M))$.

In fact, it is worth saying that our paper [1] (in particular, the relation between the X_M-cohomology and X_M-harmonic fields) can be used to recover most of the results in chapter three of [14] on $\Omega^{\pm}_G(M)$ but in terms of the operators d_{X_M}, δ_{X_M} and Δ_{X_M}. In this paper we will need the following theorem which can be proved by using the X_M-Hodge-Morrey-Friedrichs decompositions (1.5).

Theorem 2.8 *Let M be a compact, oriented smooth Riemannian manifold of dimension n with boundary and with an action of a torus G which acts by isometries on M. Given $\chi, \rho \in \Omega^{\mp}_G(M)$ and $\psi \in \Omega^{\pm}_G(\partial M)$, the boundary value problem*

$$d_{X_M}\omega = \chi \quad \textit{and} \quad \delta_{X_M}\omega = \rho \quad \textit{on} \quad M$$
$$i^*\omega = \psi \quad \textit{on} \quad \partial M \tag{2.3}$$

is solvable for $\omega \in \Omega^{\pm}_G(M)$ *if and only if the data obey the integrability conditions*

$$\delta_{X_M}\rho = 0, \quad \langle \rho, \kappa \rangle = 0, \quad \forall \kappa \in \mathcal{H}^{\mp}_{X_M,D}(M) \tag{2.4}$$

and

$$d_{X_M}\chi = 0, \quad i^*\chi = d_{X_M}\psi, \quad \langle \chi, \kappa \rangle = \int_{\partial M} \psi \wedge i^* \star \kappa, \quad \forall \kappa \in \mathcal{H}^{\mp}_{X_M,D}(M) \tag{2.5}$$

The solution of eq.(2.3) is unique up to arbitrary Dirichlet X_M*- harmonic fields* $\kappa \in \mathcal{H}^{\pm}_{X_M,D}(M)$

PROOF: The proof is analogous to the proof of theorem 3.2.5 of [14] but in terms of the operators d_{X_M} and δ_{X_M} . ❒

Lemma 2.9

$$i^*\mathcal{H}^{\pm}_{X_M}(M) = \mathcal{E}^{\pm}_{X_M}(\partial M) + i^*\mathcal{H}^{\pm}_{X_M,N}(M) \tag{2.6}$$

where $\mathcal{E}^{\pm}_{X_M}(\partial M) = \{d_{X_M}\alpha \mid \alpha \in \Omega^{\mp}_G(\partial M)\}$

PROOF: We first prove that, $i^*\mathcal{H}^{\pm}_{X_M}(M) \subseteq \mathcal{E}^{\pm}_{X_M}(\partial M) + i^*\mathcal{H}^{\pm}_{X_M,N}(M)$.

Suppose $\lambda \in \mathcal{H}^{\pm}_{X_M}(M)$ then the X_M-Friedrichs Decomposition theorem (1.4) implies that

$$\lambda = \mathsf{d}_{X_M}\alpha + \lambda_N \in \mathcal{H}^{\pm}_{X_M,N}(M) \oplus \mathcal{H}^{\pm}_{X_M,\mathrm{ex}}(M)$$

Hence,

$$i^*\lambda = \mathsf{d}_{X_M} i^*\alpha + i^*\lambda_N.$$

Conversely, it is clear that $i^*\mathcal{H}^{\pm}_{X_M,N}(M) \subseteq i^*\mathcal{H}^{\pm}_{X_M}(M)$. So, we only need to prove that $\mathcal{E}^{\pm}_{X_M}(\partial M) \subseteq i^*\mathcal{H}^{\pm}_{X_M}(M)$. Suppose, $\eta = d_{X_M}\alpha \in \mathcal{E}^{\pm}_{X_M}(\partial M)$ then η satisfies

$$d_{X_M}\eta = 0, \quad \int_{\partial M} d_{X_M}\alpha \wedge i^* \star \kappa = 0, \quad \forall \kappa \in \mathcal{H}^{\mp}_{X_M,D}(M) \tag{2.7}$$

Clearly, theorem 2.8 asserts that the condition (2.7) is a necessary and sufficient condition for the existence of $\lambda \in \mathcal{H}^{\pm}_{X_M}(M)$ such that $\eta = i^*\lambda$. ❒

Remark 2.10 In [1], we define the spaces

$$\mathcal{H}^{\pm}_{X_M,\mathrm{co}}(M) = \{\eta \in \mathcal{H}^{\pm}_{X_M}(M) \mid \eta = \delta_{X_M}\alpha\}, \qquad \mathcal{H}^{\pm}_{X_M,\mathrm{ex}}(M) = \{\xi \in \mathcal{H}^{\pm}_{X_M}(M) \mid \xi = d_{X_M}\sigma\}$$

and our proof of the X_M-Friedrichs Decomposition Theorem (1.3 and 1.4) shows that the differential forms α and σ can be chosen to be X_M-closed (i.e. $d_{X_M}\alpha = 0$) and X_M-coclosed (i.e. $\delta_{X_M}\sigma = 0$) respectively and in both cases α and σ should be X_M-harmonic forms (i.e. $\Delta_{X_M}\alpha = \Delta_{X_M}\sigma = 0$). This observation will be used in section 4.

3 X_M-DN operator

Before defining this operator, we first need to prove the solvability of a certain boundary value problem BVP (3.1) which is shown in theorem 3.1. This theorem represents the keystone to define the X_M-DN operator and then to exploiting a connection between this X_M-DN operator and X_M-cohomology via the Neumann X_M-trace space $i^*\mathcal{H}^{\pm}_{X_M,N}(M)$.

Theorem 3.1 *Let M be a compact, oriented smooth Riemannian manifold of dimension n with boundary and with an action of a torus G which acts by isometries on M. Given $\theta \in \Omega^{\pm}_G(\partial M)$ and $\eta \in \Omega^{\pm}_G(M)$, then the* BVP

$$\begin{cases} \Delta_{X_M}\omega &= \eta \quad \textit{on} \quad M \\ i^*\omega &= \theta \quad \textit{on} \quad \partial M \\ i^*(\delta_{X_M}\omega) &= 0 \quad \textit{on} \quad \partial M. \end{cases} \tag{3.1}$$

is solvable for $\omega \in \Omega^{\pm}_G(M)$ if and only if

$$\langle \eta, \kappa_D \rangle = 0, \quad \forall \kappa_D \in \mathcal{H}^{\pm}_{X_M,D}(M) \tag{3.2}$$

The solution of BVP *(3.1) is unique up to an arbitrary Dirichlet X_M-harmonic field $\mathcal{H}^{\pm}_{X_M,D}(M)$.*

PROOF: Suppose eq.(3.1) has a solution then one can easily show that the condition (3.2) holds by using Green's formula (1.6).

Now, suppose the condition $\langle \eta, \kappa_D \rangle = 0, \quad \forall \kappa_D \in \mathcal{H}^{\pm}_{X_M,D}(M)$ is given (i.e. $\eta \in \mathcal{H}^{\pm}_{X_M,D}(M)^{\perp}$). Since $\theta \in \Omega^{\pm}_G(\partial M)$, we can construct an extension $\omega_1 \in \Omega^{\pm}_G(M)$ of the differential form $\theta \in \Omega^{\pm}_G(\partial M)$ such that

$$i^*\omega_1 = \theta, \quad \omega_1 = \delta_{X_M}\beta_{\omega_1} + \lambda_{\omega_1} \in \mathcal{C}^{\pm}_{X_M}(M) \oplus \mathcal{H}^{\pm}_{X_M}(M).$$

But $\Delta_{X_M}\omega_1 = \delta_{X_M} d_{X_M} \delta_{X_M}\beta_{\omega_1}$, then (1.6) implies that $\Delta_{X_M}\omega_1 \in \mathcal{H}^{\pm}_{X_M,D}(M)^{\perp}$ as well. Hence, $\eta - \Delta_{X_M}\omega_1 \in \mathcal{H}^{\pm}_{X_M,D}(M)^{\perp}$. We now apply proposition 3.8 of [1] which for smooth invariant forms states that for each $\overline{\eta} \in \mathcal{H}^{\pm}_{X_M,D}(M)^{\perp}$ there is a unique smooth differential form $\overline{\omega} \in \Omega^{\pm}_{G,D} \cap \mathcal{H}^{\pm}_{X_M,D}(M)^{\perp}$ satisfying the BVP (3.1) but with $\eta = \overline{\eta}$ and $\theta = 0$. Since $\eta - \Delta_{X_M}\omega_1 \in \mathcal{H}^{\pm}_{X_M,D}(M)^{\perp}$ is smooth, it follows from this there is a unique smooth differential form $\omega_2 \in \Omega^{\pm}_{G,D} \cap \mathcal{H}^{\pm}_{X_M,D}(M)^{\perp}$ which satisfies the BVP

$$\begin{cases} \Delta_{X_M}\omega_2 &= \eta - \Delta_{X_M}\omega_1 \quad \text{on} \quad M \\ i^*\omega_2 &= 0 \quad \text{on} \quad \partial M \\ i^*(\delta_{X_M}\omega_2) &= 0 \quad \text{on} \quad \partial M. \end{cases} \tag{3.3}$$

Now, let $\omega_2 = \omega - \omega_1$, then the BVP (3.3) turns into the BVP (3.1). Hence, there exists a solution to the BVP (3.1) which is $\omega = \omega_1 + \omega_2$, where the uniqueness of ω is up to an arbitrary Dirichlet X_M-harmonic fields. ❒

Definition 3.2 (X_M-DN operator Λ_{X_M}) Let M be the manifold in question. We consider the BVP (3.1) with $\eta = 0$, i.e.

$$\begin{cases} \Delta_{X_M}\omega &= 0 \quad \text{on} \quad M \\ i^*\omega &= \theta \quad \text{on} \quad \partial M \\ i^*(\delta_{X_M}\omega) &= 0 \quad \text{on} \quad \partial M \end{cases} \tag{3.4}$$

then the BVP (3.4) is solvable and the solution is unique up to an arbitrary Dirichlet X_M-harmonic field $\kappa_D \in \mathcal{H}^{\pm}_{X_M,D}(M)$ (Theorem 3.1). We can therefore define the X_M-DN operator $\Lambda_{X_M} : \Omega^{\pm}_G(\partial M) \longrightarrow$

$\Omega_G^{n-(\mp)}(\partial M)$ by

$$\Lambda_{X_M}\theta = i^*(\star d_{X_M}\omega).$$

Note that taking $d_{X_M}\omega$ eliminates the ambiguity in the choice of the solution ω which means $\Lambda_{X_M}\theta$ is well defined.

In the case of $X_M = 0$, the definition (3.2) reduces to the definition of Belishev and Sharafutdinov's DN-operator Λ [6].

The remainder of our results in this section are slightly the analogues of the results in [6].

Lemma 3.3 *Let $\omega \in \Omega_G^{\pm}(M)$ be a solution to the* BVP *(3.4) where $\theta \in \Omega_G^{\pm}(\partial M)$ is given. Then $d_{X_M}\omega \in \mathcal{H}_{X_M}^{\mp}(M)$ and $\delta_{X_M}\omega = 0$.*

PROOF: Since d_{X_M} commutes with i^* and Δ_{X_M} then the BVP (3.4) and $\Lambda_{X_M}\theta = i^*(\star d_{X_M}\omega)$ shows that $d_{X_M}\omega$ solves the BVP

$$\Delta_{X_M} d_{X_M}\omega = 0, \quad i^*(\star d_{X_M}^2\omega) = 0, \quad i^*(\delta_{X_M} d_{X_M}\omega) = 0.$$

But proposition 3.2(4) of [1] implies that $d_{X_M}\omega \in \mathcal{H}_{X_M}^{\mp}(M)$.

Since $d_{X_M}\omega \in \mathcal{H}_{X_M}^{\mp}(M)$, one can easily verify that $\mathrm{d}_{X_M}\delta_{X_M}\omega = -\delta_{X_M}\mathrm{d}_{X_M}\omega = 0$ and $\delta_{X_M}^2\omega = 0$ which means that $\delta_{X_M}\omega \in \mathcal{H}_{X_M,\mathrm{co}}^{\pm}(M)$ but the second condition (i.e. $i^*(\delta_{X_M}\omega) = 0$) of the BVP (3.4) gives that $\delta_{X_M}\omega \in \mathcal{H}_{X_M,D}^{\pm}(M)$. Using (1.3), this then implies that $\delta_{X_M}\omega \in \mathcal{H}_{X_M,D}^{\pm}(M) \cap \mathcal{H}_{X_M,\mathrm{co}}^{\pm}(M) = \{0\}$, i.e. $\delta_{X_M}\omega = 0$. ❒

Lemma 3.4 *The operator Λ_{X_M} is nonnegative in the sense that the integral*

$$\int_{\partial M} \theta \wedge \Lambda_{X_M}\theta$$

is nonnegative for any $\theta \in \Omega_G^{\pm}(\partial M)$.

PROOF: For given θ, let $\omega \in \Omega_G^{\pm}(M)$ be a solution to the BVP (3.4). Then it follows from (1.6) that

$$0 = \langle \Delta_{X_M}\omega, \omega\rangle = \langle \mathrm{d}_{X_M}\omega, \mathrm{d}_{X_M}\omega\rangle + \langle \delta_{X_M}\omega, \delta_{X_M}\omega\rangle - \int_{\partial M} i^*\omega \wedge i^*(\star \mathrm{d}_{X_M}\omega)$$

whence

$$\int_{\partial M} \theta \wedge \Lambda_{X_M}\theta = \|\mathrm{d}_{X_M}\omega\|^2 + \|\delta_{X_M}\omega\|^2 \geq 0. \tag{3.5}$$

❒

Lemma 3.5

$$\ker \Lambda_{X_M} = \operatorname{Ran} \Lambda_{X_M} = i^*\mathcal{H}_{X_M}(M)$$

where $\mathcal{H}_{X_M} = \mathcal{H}_{X_M}^{+} \oplus \mathcal{H}_{X_M}^{-}$

PROOF: We first prove that $\ker \Lambda_{X_M} = i^*\mathcal{H}_{X_M}(M)$. If $\theta = i^*\lambda \in i^*\mathcal{H}_{X_M}(M)$ for $\lambda \in \mathcal{H}_{X_M}(M)$, then λ is a solution to the BVP (3.4). But $d_{X_M}\lambda = \delta_{X_M}\lambda = 0$, therefore $\Lambda_{X_M}\theta = i^*(\star d_{X_M}\lambda) = 0$. Conversely, if $\theta \in \ker \Lambda_{X_M}$ and λ is a solution to the BVP (3.4) then $\theta = i^*\lambda$ and equation (3.5) implies that $d_{X_M}\lambda = \delta_{X_M}\lambda = 0$. i.e. $\theta = i^*\lambda \in i^*\mathcal{H}_{X_M}(M)$. Hence, $\ker \Lambda_{X_M} = i^*\mathcal{H}_{X_M}(M)$.

Now, to prove $\operatorname{Ran}\Lambda_{X_M} = i^*\mathcal{H}_{X_M}(M)$, suppose $\phi \in \operatorname{Ran}\Lambda_{X_M}$ then $\phi = \Lambda_{X_M}\theta$ where $\theta = i^*\lambda$ such that λ is a solution of the BVP (3.4). But, $d_{X_M}\lambda \in \mathcal{H}_{X_M}(M)$ (Lemma 3.3) then $\star d_{X_M}\lambda \in \mathcal{H}_{X_M}(M)$ too. Hence, $\phi = \Lambda_{X_M}\theta = i^*(\star d_{X_M}\lambda) \in i^*\mathcal{H}_{X_M}(M)$. Conversely, let $\phi = i^*\lambda \in i^*\mathcal{H}_{X_M}(M)$, i.e. $\lambda \in \mathcal{H}_{X_M}(M)$. Applying, the X_M-Friedrichs Decomposition Theorem (1.4), we can decompose $\star\lambda$ as

$$\star\lambda = \mathsf{d}_{X_M}\omega + \lambda_N \in \mathcal{H}_{X_M,N}(M) \oplus \mathcal{H}_{X_M,\mathrm{ex}}(M). \tag{3.6}$$

Remark 2.10 asserts that ω can be chosen such that

$$\Delta_{X_M}\omega = 0, \quad \delta_{X_M}\omega = 0$$

which implies that

$$\Lambda_{X_M} i^*\omega = i^*(\star d_{X_M}\omega).$$

We can obtain from eq. (3.6) that

$$i^*(\star d_{X_M}\omega) = \pm i^*\lambda.$$

Comparing the last two equation with $\phi = i^*\lambda$, we obtain $\phi = \Lambda_{X_M}(\pm i^*\omega) \in \operatorname{Ran}\Lambda_{X_M}$.

❒

Corollary 3.6 *The operator Λ_{X_M} satisfies the following relations:*

$$\Lambda_{X_M}\mathsf{d}_{X_M} = 0, \quad \mathsf{d}_{X_M}\Lambda_{X_M} = 0, \quad \Lambda^2_{X_M} = 0. \tag{3.7}$$

PROOF: The first relation of (3.7) means that any form in the space $\mathcal{E}_{X_M}(\partial M)$ is the trace of an X_M-harmonic field which is true by $\mathcal{E}_{X_M}(\partial M) \subseteq i^*\mathcal{H}_{X_M}(M) = \ker\Lambda_{X_M}$ (Lemmas 2.9 and 3.5) while the second and third of equalities (3.7) follow from Lemma 3.5. ❒

Corollary 3.7 *The operator $\mathsf{d}_{X_M}\Lambda^{-1}_{X_M} : i^*\mathcal{H}_{X_M}(M) \longrightarrow i^*\mathcal{H}_{X_M}(M)$ is well-defined, i.e. the equation $\phi = \Lambda_{X_M}\theta$ has a solution θ for any $\phi \in i^*\mathcal{H}_{X_M}(M)$, and $\mathsf{d}_{X_M}\theta$ is uniquely determined by $\phi = \Lambda_{X_M}\theta$. In particular, the operator $\mathsf{d}_{X_M}\Lambda^{-1}_{X_M}\mathsf{d}_{X_M} : \Omega_G(\partial M) \longrightarrow \Omega_G(\partial M)$ is well-defined.*

PROOF: Lemma 3.5 proves that $\operatorname{Ran}\Lambda_{X_M} = i^*\mathcal{H}_{X_M}(M)$. Hence, if $\phi \in i^*\mathcal{H}_{X_M}(M)$ then the equation $\phi = \Lambda_{X_M}\theta$ is solvable. If $\Lambda_{X_M}\theta_1 = \Lambda_{X_M}\theta_2$ then $\theta_1 - \theta_2 \in \ker\Lambda_{X_M}$ is X_M-closed (i.e. $\mathsf{d}_{X_M}(\theta_1 - \theta_2) = 0$) because $\ker\Lambda_{X_M} = i^*\mathcal{H}_{X_M}(M)$. Thus, $\mathsf{d}_{X_M}\theta_1 = \mathsf{d}_{X_M}\theta_2$ which means that $\mathsf{d}_{X_M}\theta$ is uniquely determined by $\phi = \Lambda_{X_M}\theta$. Clearly, the operator $\mathsf{d}_{X_M}\Lambda^{-1}_{X_M}\mathsf{d}_{X_M}$ is well-defined because we have shown in lemma 2.9 that $\mathcal{E}_{X_M}(\partial M) \subseteq i^*\mathcal{H}_{X_M}(M)$. ❒

4 Λ_{X_M} operator, X_M-cohomology and equivariant cohomology

In the following theorem which is the analogues of theorem 1.2, we relate the $\dim(H^{\pm}_{X_M}(M))$ with the kernel of Λ_{X_M} as follows:

Theorem 4.1 *Let $\Lambda^{\pm}_{X_M}$ be the restriction of X_M-DN operator to $\Omega^{\pm}_G(\partial M)$. Then $\mathcal{E}^{\pm}_{X_M}(\partial M) \subseteq \ker\Lambda^{\pm}_{X_M}$ and*

$$\dim[\ker\Lambda^{\pm}_{X_M}/\mathcal{E}^{\pm}_{X_M}(\partial M)] \leq \min\{\dim(H^{\pm}_{X_M}(\partial M)), \dim(H^{\pm}_{X_M}(M))\}$$

PROOF: We can apply the X_M-Hodge-Morrey decomposition theorem (1.2) (or theorem 2.5 of [1]) for ∂M which asserts that the direct sum of the first and third subspaces is equal to the subspace of all X_M-closed invariant differential $\pm$-forms (that is, $\ker \mathsf{d}_{X_M}$). Hence, this fact together with eq.(3.7) implies that

$$\mathcal{E}^{\pm}_{X_M}(\partial M) \subset \ker \Lambda^{\pm}_{X_M} \subset \mathcal{H}^{\pm}_{X_M}(\partial M) \oplus \mathcal{E}^{\pm}_{X_M}(\partial M).$$

This implies

$$\dim[\ker \Lambda^{\pm}_{X_M}/\mathcal{E}^{\pm}_{X_M}(\partial M)] \leq \dim \mathcal{H}^{\pm}_{X_M}(\partial M) = \dim(H^{\pm}_{X_M}(\partial M)).$$

By Lemmas 2.9 and 3.5,

$$\ker \Lambda^{\pm}_{X_M} = \mathcal{E}^{\pm}_{X_M}(\partial M) + i^*\mathcal{H}^{\pm}_{X_M,N}(M).$$

Thus,

$$\dim[\ker \Lambda^{\pm}_{X_M}/\mathcal{E}^{\pm}_{X_M}(\partial M)] \leq \dim(i^*\mathcal{H}^{\pm}_{X_M,N}(M)) = \dim(H^{\pm}_{X_M}(M)).$$

Therefore

$$\dim[\ker \Lambda^{\pm}_{X_M}/\mathcal{E}^{\pm}_{X_M}(\partial M)] \leq \min\{\dim(H^{\pm}_{X_M}(\partial M)), \dim(H^{\pm}_{X_M}(M))\}$$

as required. ❒

In particular, corollary 4.4 of [1] asserts that if the set of zeros $N(X_M)$ of the corresponding vector field X_M is equal to the fixed point set F for the G-action (i.e. $N(X_M) = F$) then $\dim(H^{\pm}_{X_M}(M)) = \operatorname{rank} H^{\pm}_G(M)$ and $\dim(H^{\pm}_{X_M}(M,\partial M)) = \operatorname{rank} H^{\pm}_G(M,\partial M)$ where $H^{\pm}_G(M)$ and $H^{\pm}_G(M,\partial M)$ are absolute and relative equivariant cohomology respectively. The X_M-Poincaré-Lefschetz duality (Theorem 3.16(c) of [1]) asserts that $\operatorname{rank} H^{\pm}_G(M) = \operatorname{rank} H^{n-(\pm)}_G(M,\partial M)$. Hence, we conclude the following corollary which relates the kernel of Λ_{X_M} with the rank of the absolute and relative equivariant cohomology. In fact, we can write down some lower bounds for that rank:

Corollary 4.2 *If $N(X_M) = F$ then we have*

$$\dim[\ker \Lambda^{\pm}_{X_M}/\mathcal{E}^{\pm}_{X_M}(\partial M)] \leq min\{\operatorname{rank} H^{\pm}_G(\partial M), \operatorname{rank} H^{\pm}_G(M)\}.$$

The following theorem is the analogues of theorem 4.2 of [6] (our theorem 1.1).

Theorem 4.3 *The Neumann X_M-trace space $i^*\mathcal{H}^{n-(\mp)}_{X_M,N}(M)$ can be completely determined from our boundary data $(\partial M, \Lambda_{X_M})$ in particular,*

$$(\Lambda_{X_M} - (\mp 1)^{n+1}\mathsf{d}_{X_M}\Lambda^{-1}_{X_M}\mathsf{d}_{X_M})\Omega^{\pm}_G(\partial M) = i^*\mathcal{H}^{n-(\mp)}_{X_M,N}(M) \tag{4.1}$$

PROOF: We need first to prove that

$$(\Lambda_{X_M} - (\mp 1)^{n+1}\mathsf{d}_{X_M}\Lambda^{-1}_{X_M}\mathsf{d}_{X_M})\Omega^{\pm}_G(\partial M) \subseteq i^*\mathcal{H}^{n-(\mp)}_{X_M,N}(M)$$

Suppose $\theta \in \Omega^{\pm}_G(\partial M)$, let $\omega \in \Omega^{\pm}_G(M)$ be a solution to the BVP (3.4). Lemma (3.3) proves that $d_{X_M}\omega \in \mathcal{H}^{\mp}_{X_M}(M)$. Applying the X_M-Friedrichs decomposition to $\mathsf{d}_{X_M}\omega$, we get

$$\mathsf{d}_{X_M}\omega = \delta_{X_M}\alpha + \lambda_D \in \mathcal{H}^{\mp}_{X_M,\mathrm{co}}(M) \oplus \mathcal{H}^{\mp}_{X_M,D}(M) \tag{4.2}$$

where $\alpha \in \Omega^{\pm}_G(M)$ and by remark 2.10, α can be chosen such that

$$\mathsf{d}_{X_M}\alpha = 0, \quad \Delta_{X_M}\alpha = 0 \tag{4.3}$$

we set $\beta = \star\alpha \in \Omega^{n-\pm}_G(M)$. Hence, eq.(4.3) implies

$$\delta_{X_M}\beta = 0, \quad \Delta_{X_M}\beta = 0 \tag{4.4}$$

substituting $\alpha = (\pm 1)^{n+1}\star\beta$ into eq.(4.2), we have

$$\mathsf{d}_{X_M}\omega = (\pm 1)^{n+1}\delta_{X_M}\star\beta + \lambda_D \tag{4.5}$$

which implies

$$i^*(\mathsf{d}_{X_M}\omega) = (\pm 1)^{n+1} i^*(\delta_{X_M}\star\beta). \tag{4.6}$$

But,

$$i^*(\mathsf{d}_{X_M}\omega) = d_{X_M}(i^*\omega) = \mathsf{d}_{X_M}\theta$$

and

$$\delta_{X_M}\star\beta = \mp(-1)^n\star\mathsf{d}_{X_M}\beta$$

thus, eq.(4.6) turns into

$$\mathsf{d}_{X_M}\theta = -(\mp 1)^n i^*(\star\mathsf{d}_{X_M}\beta) \tag{4.7}$$

Formulas (4.4) and (4.7) mean that

$$d_{X_M}\theta = -(\mp 1)^n \Lambda_{X_M} i^*\beta. \tag{4.8}$$

Now, applying, ($i^*\star$) to eq.(4.5) with the fact that $\Lambda_{X_M}\theta = i^*(\star\mathsf{d}_{X_M}\omega)$, we get

$$\Lambda_{X_M}\theta = (\pm 1)^{n+1} i^*(\star\delta_{X_M}\star\beta) + i^*(\star\lambda_D). \tag{4.9}$$

Using the relation $\star\delta_{X_M}\star\beta = (\pm 1)^n\mathsf{d}_{X_M}\beta$, then eq.(4.9) reduces to

$$\Lambda_{X_M}\theta = \pm\mathsf{d}_{X_M}(i^*\beta) + i^*(\star\lambda_D) \tag{4.10}$$

we can obtain from eq.(4.8) that

$$\mathsf{d}_{X_M}(i^*\beta) = -(\mp 1)^n d_{X_M}\Lambda^{-1}_{X_M} d_{X_M}\theta$$

Putting the latter equation in eq.(4.10), we get

$$i^*(\star\lambda_D) = (\Lambda_{X_M} - (\mp 1)^{n+1}\mathsf{d}_{X_M}\Lambda^{-1}_{X_M}\mathsf{d}_{X_M})\theta.$$

Hence, $(\Lambda_{X_M} - (\mp 1)^{n+1}\mathsf{d}_{X_M}\Lambda^{-1}_{X_M}\mathsf{d}_{X_M})\theta \in i^*\mathcal{H}^{n-(\mp)}_{X_M,N}(M)$.

The next step is then to prove the converse, i.e.

$$i^*\mathcal{H}^{n-(\mp)}_{X_M,N}(M) \subseteq (\Lambda_{X_M} - (\mp 1)^{n+1}\mathsf{d}_{X_M}\Lambda^{-1}_{X_M}\mathsf{d}_{X_M})\Omega^{\pm}_G(\partial M)$$

Given $\lambda_N \in \mathcal{H}^{n-(\mp)}_{X_M,N}(M)$, then corollary 2.4 asserts that λ_N has the following representation

$$\lambda_N = \mathsf{d}_{X_M}\alpha + \delta_{X_M}\beta \in \mathcal{H}^{n-\mp}_{X_M,\mathrm{ex}}(M) + \mathcal{H}^{n-\mp}_{X_M,\mathrm{co}}(M) \tag{4.11}$$

and also by remark 2.10, α and β can be chosen respectively to satisfy

$$\delta\alpha = 0, \quad \Delta_{X_M}\alpha = 0 \tag{4.12}$$

and

$$\mathsf{d}_{X_M}\beta = 0, \quad \Delta_{X_M}\beta = 0 \tag{4.13}$$

We set up the transformations

$$\omega = -(\pm 1)^n \star \beta, \quad \varepsilon = -(\mp 1)^{n+1}\alpha$$

Then eqs.(4.12)-(4.13) turn into

$$\delta\omega = 0, \quad \Delta_{X_M}\omega = 0 \tag{4.14}$$

$$\delta_{X_M}\varepsilon = 0, \quad \Delta_{X_M}\varepsilon = 0 \tag{4.15}$$

and eq.(4.11) implies

$$\lambda_N = \star\mathsf{d}_{X_M}\omega - (\mp 1)^{n+1}\mathsf{d}_{X_M}\varepsilon \tag{4.16}$$

hence,

$$\star\lambda_N = -(\mp 1)^{n+1}(\star\mathsf{d}_{X_M}\varepsilon - \mathsf{d}_{X_M}\omega). \tag{4.17}$$

We can define forms $\phi, \psi \in \Omega_G(\partial M)$ by setting

$$\phi = i^*\omega, \quad \psi = i^*\varepsilon \tag{4.18}$$

Restricting eq.(4.16) to the boundary and using the fact that $i^* \star \mathsf{d}_{X_M}\omega = \Lambda_{X_M}\phi$, we obtain

$$i^*\lambda_N = \Lambda_{X_M}\phi - (\mp 1)^{n+1}\mathsf{d}_{X_M} i^*\varepsilon \tag{4.19}$$

Restricting eq.(4.17) to the boundary

$$i^*(\star\mathsf{d}_{X_M}\varepsilon) = \mathsf{d}_{X_M}(i^*\omega) \tag{4.20}$$

but $i^*(\star\mathsf{d}_{X_M}\varepsilon) = \Lambda_{X_M}\psi$ because of eq.(4.15) and the second of equality (4.18). Hence, eq.(4.20) turns to

$$\Lambda_{X_M}\psi = \mathsf{d}_{X_M}\phi \tag{4.21}$$

Now, we can eliminate the form ψ from eq.(4.19) and eq.(4.21) and we can obtain that

$$i^*\lambda_N = (\Lambda_{X_M} - (\mp 1)^{n+1}\mathsf{d}_{X_M}\Lambda_{X_M}^{-1}\mathsf{d}_{X_M})\phi$$

Hence, $i^*\lambda_N \in (\Lambda_{X_M} - (\mp 1)^{n+1}\mathsf{d}_{X_M}\Lambda_{X_M}^{-1}\mathsf{d}_{X_M})\Omega_G^{\pm}(\partial M)$. ❒

5 X_M- Hilbert transform

In this section, we introduce the X_M- Hilbert transform which will be used in section 6. We begin with the following definition.

Definition 5.1 (X_M- Hilbert transform) The X_M- Hilbert transform is the operator

$$T_{X_M} = \mathsf{d}_{X_M}\Lambda_{X_M}^{-1} : i^*\mathcal{H}_{X_M}^{\pm}(M) \longrightarrow i^*\mathcal{H}_{X_M}^{n-(\pm)}(M).$$

T_{X_M} is a well-defined operator by corollary 3.7 and the restriction of T_{X_M} to X_M-exact boundary forms $\mathcal{E}_{X_M}^{\pm}(\partial M) \subseteq i^*\mathcal{H}_{X_M}^{\pm}(M)$ satisfies

$$T_{X_M} : \mathcal{E}_{X_M}^{\pm}(\partial M) \longrightarrow \mathcal{E}_{X_M}^{n-(\pm)}(\partial M).$$

Lemma 5.2 *The X_M- Hilbert transform maps $i^*\mathcal{H}^{\pm}_{X_M,N}(M)$ to $i^*\mathcal{H}^{n-(\pm)}_{X_M,N}(M)$.*

PROOF: Let $\varphi \in i^*\mathcal{H}^{\pm}_{X_M,N}(M)$ then theorem 4.3 implies that

$$\varphi = (\Lambda_{X_M} - (\pm 1)^{n+1}\mathsf{d}_{X_M}\Lambda^{-1}_{X_M}\mathsf{d}_{X_M})\theta$$

for some $\theta \in \Omega^{n-(\mp)}(\partial M)$. Hence, it follows that

$$\begin{aligned} T_{X_M}\varphi &= \mathsf{d}_{X_M}\Lambda^{-1}_{X_M}(\Lambda_{X_M} - (\pm 1)^{n+1}\mathsf{d}_{X_M}\Lambda^{-1}_{X_M}\mathsf{d}_{X_M})\theta \\ &= (\mathsf{d}_{X_M} - (\pm 1)^{n+1}\mathsf{d}_{X_M}\Lambda^{-1}_{X_M}\mathsf{d}_{X_M}\Lambda^{-1}_{X_M}\mathsf{d}_{X_M})\theta \\ &= (\Lambda_{X_M} - (\pm 1)^{n+1}\mathsf{d}_{X_M}\Lambda^{-1}_{X_M}\mathsf{d}_{X_M})\Lambda^{-1}_{X_M}\mathsf{d}_{X_M}\theta \\ &= (\Lambda_{X_M} - (\pm 1)^{n+1}\mathsf{d}_{X_M}\Lambda^{-1}_{X_M}\mathsf{d}_{X_M})\Lambda^{-1}_{X_M}\mathsf{d}_{X_M}\theta \end{aligned}$$

but $\Lambda^{-1}_{X_M}\mathsf{d}_{X_M}(\theta) \in \Omega^{\mp}_G(\partial M)$. Thus, by theorem (4.3) we find that the right hand side of the latter formula must belong to $i^*\mathcal{H}^{n-(\pm)}_{X_M,N}(M)$. ❐

6 Recovering X_M-cohomology from the boundary data $(\partial M, \Lambda_{X_M})$

In this section we pose two questions where in subsection 6.1 we present our answer to the following first question:

"Can the additive structure of the real absolute and relative X_M-cohomology be completely recovered from the boundary data $(\partial M, \Lambda_{X_M})$?." The answer is affirmative and more precisely, we show that the data $(\partial M, \Lambda_{X_M})$ determines the long exact sequence of X_M-cohomology of the topological pair $(M, \partial M)$.

While in subsection 6.2, we present a partial answer to the following second question:

"Can the ring (i.e. multiplicative) structure of the real absolute and relative X_M-cohomology be completely recovered from the boundary data $(\partial M, \Lambda_{X_M})$?"

6.1 Recovering the additive real X_M-cohomology.

Since the vector field X_M which we are considering is always tangent to the boundary ∂M then we can still define X_M-cohomology on ∂M, i.e. $H^{\pm}_{X_M}(\partial M)$. Hence, from our definitions of the absolute and relative X_M-cohomology [1], we can set up the following exact X_M-cohomology sequence of the pair $(M, \partial M)$ as follows:

$$\dots \xrightarrow{\pi^*} H^{\pm}_{X_M}(M,\partial M) \xrightarrow{\rho^*} H^{\pm}_{X_M}(M) \xrightarrow{i^*} H^{\pm}_{X_M}(\partial M) \xrightarrow{\pi^*} H^{\mp}_{X_M}(M,\partial M) \xrightarrow{\rho^*} \dots \qquad (6.1)$$

where

1. $i^*[\omega]_{(X_M,M)} = [i^*\omega]_{(X_M,\partial M)}, \quad \forall [\omega]_{(X_M,M)} \in H^{\pm}_{X_M}(M)$.
2. $\rho^*[\omega]_{(X_M,M,\partial M)} = [\omega]_{(X_M,M)}, \quad \forall [\omega]_{(X_M,M,\partial M)} \in H^{\pm}_{X_M}(M,\partial M)$. In fact, the operator ρ^* is induced by the embedding of pairs $\rho : (M, \emptyset) \subset (M, \partial M)$. ρ^* is well-defined.
3. $\pi^*[\omega]_{(X_M,\partial M)} = [\mathsf{d}_{X_M}\alpha]_{(X_M,M,\partial M)}, \quad \forall [\omega]_{(X_M,\partial M)} \in H^{\pm}_{X_M}(\partial M)$, where $\alpha \in \Omega^{\pm}_G(M)$ is any extension of $\omega \in \Omega^{\pm}_G(\partial M)$ to M, i.e. $i^*\alpha = \omega$. Since d_{X_M} and i^* commute, then $[\mathsf{d}_{X_M}\alpha]_{(X_M,M,\partial M)} \in H^{\mp}_{X_M}(M,\partial M)$. The form $\mathsf{d}_{X_M}\alpha$ is certainly X_M-exact, but is not in general relatively X_M-exact, i.e. $i^*\alpha \neq 0$.

Sequence (6.1) is exact in the sense that at each stage the image of the incoming homomorphism is the kernel of the outgoing one.

Now, to answer the above first question, we use theorem 4.3 which shows that we can determine the space $i^*\mathcal{H}^{\pm}_{X_M,N}(M)$ from our boundary data and corollary 2.6 which gives us the isomorphisms f and h.

So, if the boundary data $(\partial M, \Lambda_{X_M})$ is given then we can construct the sequence

$$\cdots \xrightarrow{\overline{\pi}^*} i^*\mathcal{H}^{n-(\pm)}_{X_M,N}(M) \xrightarrow{\overline{\rho}^*} i^*\mathcal{H}^{\pm}_{X_M,N}(M) \xrightarrow{\overline{i}^*} H^{\pm}_{X_M}(\partial M) \xrightarrow{\overline{\pi}^*} i^*\mathcal{H}^{n-(\mp)}_{X_M,N}(M) \xrightarrow{\overline{\rho}^*} \cdots \tag{6.2}$$

where we define the operators of sequence (6.2) by the following formulas:

1. $\overline{i}^*\theta = [\theta]_{(X_M,\partial M)}, \quad \forall \theta \in i^*\mathcal{H}^{\pm}_{X_M,N}$. i.e. $\theta = i^*\omega$ where $\omega \in \mathcal{H}^{\pm}_{X_M,N}$, then θ is X_M-closed because i^* and d_{X_M} commute.

2. Using Lemma 5.2, then we set
$$\overline{\rho}^*\theta = -(\pm 1)^{n+1} T_{X_M}\theta, \quad \forall \theta \in i^*\mathcal{H}^{n-(\pm)}_{X_M,N}$$

3. Based on theorem 4.3, then $\Lambda_{X_M}\theta = (\Lambda_{X_M} - (\mp 1)^{n+1}\mathsf{d}_{X_M}\Lambda^{-1}_{X_M}\mathsf{d}_{X_M})\theta$ if $[\theta]_{(X_M,\partial M)} \in H^{\pm}_{X_M}(\partial M)$. Hence, we set
$$\overline{\pi}^*[\theta]_{(X_M,\partial M)} = (\mp 1)^{n+1}\Lambda_{X_M}\theta, \quad \forall \in [\theta]_{(X_M,\partial M)} \in H^{\pm}_{X_M}(\partial M).$$

More concretely, our goal is then to recover sequence (6.1) from sequence (6.2). It means that we should prove that the following diagram (6.3) is commutative diagram.

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\overline{\pi}^*} & i^*\mathcal{H}^{n-(\pm)}_{X_M,N}(M) & \xrightarrow{\overline{\rho}^*} & i^*\mathcal{H}^{\pm}_{X_M,N}(M) & \xrightarrow{\overline{i}^*} & H^{\pm}_{X_M}(\partial M) & \xrightarrow{\overline{\pi}^*} & i^*\mathcal{H}^{n-(\mp)}_{X_M,N}(M) & \xrightarrow{\overline{\rho}^*} & \cdots \\
 & & \Big\downarrow h & & \Big\downarrow f & & \Big\downarrow \iota & & \Big\downarrow h & & \\
\cdots & \xrightarrow{\pi^*} & H^{\pm}_{X_M}(M,\partial M) & \xrightarrow{\rho^*} & H^{\pm}_{X_M}(M) & \xrightarrow{i^*} & H^{\pm}_{X_M}(\partial M) & \xrightarrow{\pi^*} & H^{\mp}_{X_M}(M,\partial M) & \xrightarrow{\rho^*} & \cdots
\end{array} \tag{6.3}$$

where ι is the identity operator. But, one can prove the commutativity of the diagram (6.3) by a method similar to that given in [6] but in terms of our operators above.

Actually, the above construction proves that the data $(\partial M, \Lambda_{X_M})$ recovers sequence (6.1) of the pair $(M, \partial M)$ up to an isomorphism (i.e. f and h are given in corollary 2.6) from the sequence (6.2). We therefore can state the following theorem.

Theorem 6.1 *The boundary data $(\partial M, \Lambda_{X_M})$ completely determines the additive real absolute and relative X_M-cohomology structure by showing the diagram (6.3) is commutative and then*

$$H^{\pm}_{X_M}(M) \cong (\Lambda_{X_M} - (\pm 1)^{n+1}\mathsf{d}_{X_M}\Lambda^{-1}_{X_M}\mathsf{d}_{X_M})\Omega^{n-(\mp)}_G(\partial M) \tag{6.4}$$
$$H^{\pm}_{X_M}(M,\partial M) \cong (\Lambda_{X_M} - (\pm 1)^{n+1}\mathsf{d}_{X_M}\Lambda^{-1}_{X_M}\mathsf{d}_{X_M})\Omega^{\mp}_G(\partial M) \tag{6.5}$$

6.2 Recovering the ring structure of the real X_M-cohomology.

First of all, we consider the mixed cup product $\overline{\cup}$ between the absolute and relative X_M-cohomology as follows:

$$\overline{\cup}: H^{\pm}_{X_M}(M) \times H^{\pm}_{X_M}(M,\partial M) \longrightarrow H^{\pm}_{X_M}(M,\partial M)$$

by setting

$$[\alpha]_{(X_M,M)}\overline{\cup}[\beta]_{(X_M,M,\partial M)} = [\alpha\wedge\beta]_{(X_M,M,\partial M)}, \quad \forall([\alpha]_{(X_M,M)},[\beta]_{(X_M,M,\partial M)}) \in H^{\pm}_{X_M}(M)\times H^{\pm}_{X_M}(M,\partial M)$$

it is easy to check that $\overline{\cup}$ is a well-defined map. In addition, corollary 3.17 of [1] asserts that any absolute and relative X_M-cohomology classes contain a unique Neumann and Dirichlet X_M-harmonic field respectively. Hence, we can regard any absolute (relative) X_M-cohomology class as a Neumann(Dirichlet) X_M-harmonic field. But $[\alpha]_{(X_M,M)}\overline{\cup}[\beta]_{(X_M,M,\partial M)} = [\alpha\wedge\beta]_{(X_M,M,\partial M)}$ is a relative X_M-cohomology class, so there exists a unique Dirichlet X_M-harmonic field $\eta \in \mathcal{H}^{\pm}_{X_M,D}(M)$ such that $[\alpha\wedge\beta]_{(X_M,M,\partial M)} = [\eta]_{(X_M,M,\partial M)}$, i.e.

$$\alpha\wedge\beta = \eta + d_{X_M}\xi \in \mathcal{H}^{\pm}_{X_M,D}(M)\oplus\mathcal{E}^{\pm}_{X_M}(M). \tag{6.6}$$

But, we can get from corollary 2.6 that

$$H^{\pm}_{X_M}(M,\partial M) \cong H^{n-(\pm)}_{X_M}(M) \cong i^*\mathcal{H}^{n-(\pm)}_{X_M,N}(M)$$

According to our illustrations above we know that an absolute X_M-cohomology class $[\alpha]_{(X_M,M)} \in H^{\pm}_{X_M}(M)$ and relative X_M-cohomology classes $[\beta]_{(X_M,M,\partial M)}, [\alpha\wedge\beta]_{(X_M,M,\partial M)} \in H^{\pm}_{X_M}(M,\partial M)$ are represented by the Neumann X_M-harmonic field $\alpha \in \mathcal{H}^{\pm}_{X_M,N}(M)$ and the Dirichlet X_M-harmonic fields $\beta,\eta \in \mathcal{H}^{\pm}_{X_M,D}(M)$ respectively, such that they correspond, respectively, to forms on the boundary by setting

$$\phi = i^*\alpha \in i^*\mathcal{H}^{\pm}_{X_M,N}(M), \quad \psi = i^*\star\beta \in i^*\mathcal{H}^{n-(\pm)}_{X_M,N}(M), \quad \vartheta = i^*\star\eta \in i^*\mathcal{H}^{n-(\pm)}_{X_M,N}(M)$$

As alluded to before, our answer to the second question will only be partial, in the sense that we will not consider all the classes of the relative X_M-cohomology. In fact, we will just consider the *boundary subspace* (which we denote by $BH^{\pm}_{X_M}(M,\partial M)$) of $H^{\pm}_{X_M}(M,\partial M)$. We define $BH^{\pm}_{X_M}(M,\partial M)$ as follows:

$$BH^{\pm}_{X_M}(M,\partial M) = \{[\mathrm{d}_{X_M}\rho] \mid \rho \in \Omega^{\mp}_G(M), i^*(\mathrm{d}_{X_M}\rho) = 0\}$$

Actually, in sequence (6.1), our definition of the operator π^* represents the definition of the boundary subspace of $H^{\pm}_{X_M}(M,\partial M)$. More precisely, the image of $H^{\pm}_{X_M}(\partial M)$ inside $H^{\mp}_{X_M}(M,\partial M)$ represents the natural portion to interpret as coming from the boundary. But, we have proved that $H^{\pm}_{X_M}(M,\partial M) \cong \mathcal{H}^{\pm}_{X_M,D}(M)$. Hence, on translation into the language of X_M-harmonic fields, we can identify

$$BH^{\pm}_{X_M}(M,\partial M) \cong \mathcal{E}\mathcal{H}^{\pm}_{X_M,D}(M)$$

where $\mathcal{E}\mathcal{H}^{\pm}_{X_M,D}(M) = \{\mathrm{d}_{X_M}\varepsilon \in \mathcal{H}^{\pm}_{X_M,D}(M) \mid \varepsilon \in \Omega^{\mp}_G(M)\}$. Clearly, Hodge star $\star$ gives

$$c\mathcal{E}\mathcal{H}^{n-\pm}_{X_M,N}(M) = \star\mathcal{E}\mathcal{H}^{\pm}_{X_M,D}(M)$$

where $c\mathcal{E}\mathcal{H}^{n-\pm}_{X_M,N}(M) = \{\delta_{X_M}\lambda \in \mathcal{H}^{n-\pm}_{X_M,N}(M) \mid \lambda \in \Omega^{n-\mp}_G(M)\}$. Now, using this fact together with corollary 2.6(2) we conclude that $BH^{\pm}_{X_M}(M,\partial M) \cong i^*\star\mathcal{E}\mathcal{H}^{\pm}_{X_M,D}(M)$.

Now, we adapt Shonkwiler's map [16] but in terms of our operators in order to define the following map with notation as above

$$\phi \overline{\cup}_{X_M} \psi = \Lambda_{X_M}(\pm\phi \wedge \Lambda_{X_M}^{-1}\psi), \quad \forall(\phi,\psi) \in i^*\mathcal{H}^{\pm}_{X_M,N}(M) \times i^*\mathcal{H}^{n-(\pm)}_{X_M,N}(M) \tag{6.7}$$

By using the same method as [16] but together with our definition 3.2 we deduce that $\overline{\cup}_{X_M}$ is well-defined.

So, our partial answer to the second question is that: restricting $H^{\pm}_{X_M}(M,\partial M)$ to $BH^{\pm}_{X_M}(M,\partial M)$ and then we recover the mixed cup product by showing the commutativity of the the following diagram.

Theorem 6.2 *The diagram*

$$\begin{array}{ccc} H^{\pm}_{X_M}(M) \times BH^{\pm}_{X_M}(M,\partial M) & \xrightarrow{\overline{\cup}} & BH^{\pm}_{X_M}(M,\partial M) \\ \Big\downarrow{\scriptstyle (f,h)} & & \Big\downarrow{\scriptstyle h} \\ i^*\mathcal{H}^{\pm}_{X_M,N}(M) \times i^*\star\mathcal{EH}^{\pm}_{X_M,D}(M) & \xrightarrow{\overline{\cup}_{X_M}} & i^*\star\mathcal{EH}^{\pm}_{X_M,D}(M) \end{array} \tag{6.8}$$

is commutative, where f and h are given in corollary 2.6.

PROOF: Our goal is then to show that $\forall([\alpha],[\mathsf{d}_{X_M}\beta_1]) \in H^{\pm}_{X_M}(M) \times BH^{\pm}_{X_M}(M,\partial M)$ then

$$(h\circ\overline{\cup})([\alpha],[\mathsf{d}_{X_M}\beta_1]) = (\overline{\cup}_{X_M}\circ(f,h))([\alpha],[\mathsf{d}_{X_M}\beta_1]). \tag{6.9}$$

Using eq.(6.6), then the left-hand side gives

$$\begin{aligned} h(\overline{\cup}([\alpha],[\mathsf{d}_{X_M}\beta_1])) &= h([\alpha\wedge\mathsf{d}_{X_M}\beta_1]) \\ &= h([\mathsf{d}_{X_M}(\pm\alpha\wedge\beta_1)]) \\ &= h([\mathsf{d}_{X_M}(\pm\alpha\wedge\beta_1-\xi)]) \\ &= i^*\star\eta \end{aligned} \tag{6.10}$$

while the right-hand side gives

$$\begin{aligned} \overline{\cup}_{X_M}((f([\alpha]),h([\mathsf{d}_{X_M}\beta_1]))) &= \overline{\cup}_{X_M}(i^*\alpha, i^*\star\mathsf{d}_{X_M}\beta_1) \\ &= \Lambda_{X_M}(\pm\phi\wedge\Lambda_{X_M}^{-1}\psi) \end{aligned} \tag{6.11}$$

where $\phi = i^*\alpha$ and $\psi = i^*\star\mathsf{d}_{X_M}\beta_1$. Now, we only need to show that eq.(6.10) and eq.(6.11) are equal.

Putting, $\beta = \mathsf{d}_{X_M}\beta_1 \in \mathcal{EH}^{\pm}_{X_M,D}(M)$ and using the X_M-Hodge-Morrey decomposition theorem (1.2), we infer that β_1 can be chosen to solve the BVP

$$\Delta_{X_M}\nu = 0, \quad i^*\nu = i^*\beta_1, \quad i^*\delta_{X_M}\nu = 0.$$

Hence,

$$\psi = i^*\star\mathsf{d}_{X_M}\beta_1 = \Lambda_{X_M} i^*\beta_1.$$

Therefore, $\Lambda_{X_M}^{-1}\psi = i^*\beta_1$. But from eq.(6.6) we get that

$$\eta = \mathsf{d}_{X_M}\eta' \in \mathcal{EH}^{\pm}_{X_M,D}(M)$$

where $\eta' = \pm\alpha\wedge\beta_1 - \xi$. Applying the X_M-Hodge-Morrey decomposition theorem (1.2) on η', we infer that

$$\eta = \mathsf{d}_{X_M}\eta' = \mathsf{d}_{X_M}\sigma$$

such that σ solves the BVP

$$\Delta_{X_M}\varepsilon = 0, \quad i^*\varepsilon = i^*\sigma, \quad i^*\delta_{X_M}\varepsilon = 0.$$

Hence,

$$\Lambda_{X_M} i^*\sigma = i^* \star \mathsf{d}_{X_M}\sigma = i^* \star \eta \tag{6.12}$$

Since $\eta' = \pm\alpha \wedge \beta_1 - \xi$ implies

$$\begin{aligned} \mathsf{d}_{X_M}(\pm\alpha \wedge \beta_1) &= \mathsf{d}_{X_M}\eta' + \mathsf{d}_{X_M}\xi \\ &= \mathsf{d}_{X_M}\sigma + \mathsf{d}_{X_M}\xi. \end{aligned} \tag{6.13}$$

Equation (6.13) shows that the class $[\pm\alpha \wedge \beta_1 - \sigma - \xi] \in H^{\mp}_{X_M}(M)$, so the form $\pm\alpha \wedge \beta_1 - \sigma - \xi$ can be decomposed as

$$\pm\alpha \wedge \beta_1 - \sigma - \xi = \mathsf{d}_{X_M}\tau_1 + \tau_2 \in \mathcal{E}^{\mp}_{X_M}(M) \oplus \mathcal{H}^{\mp}_{X_M}(M).$$

Now, restricting the latter equation to the boundary and using Lemma 3.5, this implies that

$$\Lambda_{X_M} i^*(\pm\alpha \wedge \beta_1 - \sigma - \xi) = \Lambda_{X_M} i^*\tau_2 = 0.$$

Combining this with equation (6.12) gives that

$$\begin{aligned} \Lambda_{X_M} i^*(\pm\alpha \wedge \beta_1) &= \Lambda_{X_M} i^*(\pm\alpha \wedge \beta_1 - \sigma - \xi + \sigma + \xi) \\ &= \Lambda_{X_M} i^*(\pm\alpha \wedge \beta_1 - \sigma - \xi) + \Lambda_{X_M} i^*\sigma \\ \Lambda_{X_M}(\pm\phi \wedge \Lambda^{-1}_{X_M}\psi) &= i^* \star \eta \end{aligned} \tag{6.14}$$

Hence, the diagram (6.8) is commutative as desired. ❒

We can restate theorem 6.2 in the language of our boundary data $(\partial M, \Lambda_{X_M})$ to be as follows:

Theorem 6.3 *The boundary data $(\partial M, \Lambda_{X_M})$ completely determines the mixed cup product structure of the X_M-cohomology when the relative X_M-cohomology classes comes from the boundary subspace. i.e. if $(\alpha, \beta) \in \mathcal{H}^{\pm}_{X_M,N}(M) \oplus \mathcal{E}\mathcal{H}^{\pm}_{X_M,D}(M)$ such that $\alpha \wedge \beta = \eta + d_{X_M}\xi \in \mathcal{H}^{\pm}_{X_M,D}(M) \oplus \mathcal{E}^{\pm}_{X_M}(M)$ then*

$$i^* \star \eta = \Lambda_{X_M}(\pm\phi \wedge \Lambda^{-1}_{X_M}\psi)$$

where $\phi = i^*\alpha$ and $\psi = i^* \star \beta$.

7 Conclusions

1- The key which uses to recover the free part of the relative and absolute equivariant cohomology groups (i.e. $H^{\pm}_G(M)$ and $H^{\pm}_G(M, \partial M)$) from our boundary data $(\partial M, \Lambda_{X_M})$ is the following theorem which has been proved in [1] based on Atiyah and Bott's localization theorem:

Theorem 7.1 ([1]) *Let $\{X_1, \ldots, X_\ell\}$ be a basis of the Lie algebra $\mathfrak{g}$ and $\{u_1, \ldots, u_\ell\}$ the corresponding coordinates and let $X = \sum_j s_j X_j \in \mathfrak{g}$. If the set of zeros $N(X_M)$ of the corresponding vector field X_M is equal to the fixed point set F for the G-action then*

$$H^{\pm}_{X_M}(M, \partial M) \cong H^{\pm}_G(M, \partial M)/\mathfrak{m}_X H^{\pm}_G(M, \partial M) \cong H^{\pm}(F, \partial F), \tag{7.1}$$

and

$$H^{\pm}_{X_M}(M) \cong H^{\pm}_G(M)/\mathfrak{m}_X H^{\pm}_G(M) \cong H^{\pm}(F) \tag{7.2}$$

where $\mathfrak{m}_X = \langle u_1 - s_1, \ldots, u_l - s_l \rangle$ is the ideal of polynomials vanishing at X.

Now, combining the above theorem with theorem 6.1, we get

Theorem 7.2 *With the hypotheses of the theorem 7.1,*

$$H^{\pm}_G(M,\partial M)/\mathfrak{m}_X H^{\pm}_G(M,\partial M) \cong H^{\pm}(F,\partial F) \cong (\Lambda_{X_M} - (\pm 1)^{n+1} \mathsf{d}_{X_M} \Lambda^{-1}_{X_M} \mathsf{d}_{X_M}) \Omega^{\mp}_G(\partial M)$$

and

$$H^{\pm}_G(M)/\mathfrak{m}_X H^{\pm}_G(M) \cong H^{\pm}(F) \cong (\Lambda_{X_M} - (\pm 1)^{n+1} \mathsf{d}_{X_M} \Lambda^{-1}_{X_M} \mathsf{d}_{X_M}) \Omega^{n-(\mp)}_G(\partial M)$$

since the Neumann X_M-harmonic fields are uniquely determined by their Neumann X_M- trace spaces (corollary 2.6) which can be completely determined by our boundary data $(\partial M, \Lambda_{X_M})$(theorem 4.3), this means that we can conclude from theorem 6.1 and theorem 7.2 that we can realize the relative and absolute X_M-cohomology groups and the free part of the relative and absolute equivariant cohomology groups as particular subspaces of invariant differential forms on ∂M and they are not just determined abstractly from our boundary data.

2- If $N(X_M) = F$ then we can apply Belishev and Sharafutdinov's results [6] (our theorem 1.1) to the manifolds F with boundary ∂F where G acts trivially on F and then we use theorem 7.2 to exploit the connection between Belishev and Sharafutdinov's boundary data on ∂F (i.e. $(\partial F, \Lambda)$) and ours on ∂M (i.e. $(\partial M, \Lambda_{X_M})$). More concretely, we have the following theorem

Theorem 7.3 *If $N(X_M) = F$, then*

$$(\Lambda_{X_M} - (\mp 1)^{n+1} \mathsf{d}_{X_M} \Lambda^{-1}_{X_M} \mathsf{d}_{X_M}) \Omega^{\pm}_G(\partial M) \cong (\Lambda - (\mp 1)^{n+1} \mathsf{d} \Lambda^{-1} \mathsf{d}) \Omega^{\pm}(\partial F).$$

3- Theorem 2.3 proves that our concrete realizations $\mathcal{H}^{\pm}_{X_M,N}(M)$ and $\mathcal{H}^{\pm}_{X_M,D}(M)$ of the absolute and relative X_M-cohomology groups inside the space $\Omega^{\pm}_G(M)$ meet only at the origin which means that we can conclude the sum $\mathcal{H}^{\pm}_{X_M,N}(M) + \mathcal{H}^{\pm}_{X_M,D}(M)$ is a direct sum and by using (1.6), we can prove that the orthogonal complement of $\mathcal{H}^{\pm}_{X_M,N}(M) + \mathcal{H}^{\pm}_{X_M,D}(M)$ inside $\mathcal{H}^{\pm}_{X_M}(M)$ is

$$\mathcal{H}^{\pm}_{X_M,\mathrm{ex}}(M) \cap \mathcal{H}^{\pm}_{X_M,\mathrm{co}}(M) = \mathcal{H}^{\pm}_{X_M,\mathrm{ex,co}}(M)$$

Therefore, we can refine our X_M-Friedrichs Decomposition (1.3 and 1.4) into

$$\mathcal{H}^{\pm}_{X_M}(M) = (\mathcal{H}^{\pm}_{X_M,N}(M) + \mathcal{H}^{\pm}_{X_M,D}(M)) \oplus \mathcal{H}^{\pm}_{X_M,\mathrm{ex,co}}(M).$$

Consequently, we can refine the X_M-Hodge-Morrey-Friedrichs decompositions (1.5) into the following five terms decomposition:

$$\Omega^{\pm}_G(M) = \mathcal{E}^{\pm}_{X_M}(M) \oplus \mathcal{C}^{\pm}_{X_M}(M) \oplus (\mathcal{H}^{\pm}_{X_M,N}(M) + \mathcal{H}^{\pm}_{X_M,D}(M)) \oplus \mathcal{H}^{\pm}_{X_M,\mathrm{ex,co}}(M).$$

The idea of this conclusion follows from [8], see also [15] for details.

Finally, it is worth considering the following important open problem:

"*Can the torsion part of the absolute and relative equivariant cohomology groups be completely recovered from our boundary data $(\partial M, \Lambda_{X_M})$?.*"

Answering this open problem will indeed complete the picture of our boundary data $(\partial M, \Lambda_{X_M})$ to be adding into the list of objects of equivariant cohomology story and consequently to the objects of algebraic topology.

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